Set Theory:

Remember: A set A is a collection of objects (elements)

x is an element of A

$x \in A$

Subsets: Let A, B are two sets.

We say A is a subset of B if

$A \subseteq B$ $\quad \forall x, (x \in A \Rightarrow x \in B)$

Example: Prove if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

Proof: Suppose z is any element of A.

Then because $A \subseteq B$, z is an elt. of B

Because $B \subseteq C$, z is a elt of C.

This is true for any z.

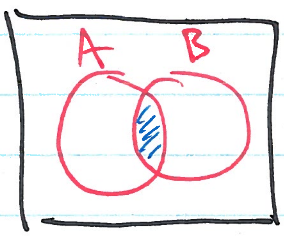
Thus $A \subseteq C$. ∎

Note: If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

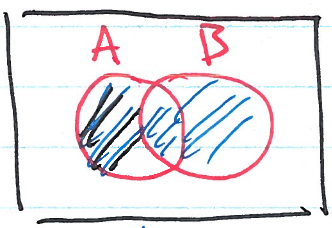
Intersection: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

Union: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

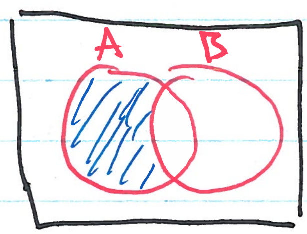
Set Difference: $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$



$A \cap B$



$A \cup B$



$A \setminus B$

Exercise: Prove that $A = (A-B) \cup (A \cap B)$

Note: We have to prove $A \subseteq (A-B) \cup (A \cap B)$

and $(A-B) \cup (A \cap B) \subseteq A$.

Proof: First we show $A \subseteq (A-B) \cup (A \cap B)$.

Suppose $x \in A$. There are two cases: either

$x \in B$ or $x \notin B$

Case 1: $x \in B$.

Then because $x \in A$ and $x \in B$, it follows $x \in A \cap B$.

Therefore, $x \in (A-B) \cup (A \cap B)$

I am using the following: if $x \in C$, then $x \in C \cup D$ for any C, D. You can use this for free.

Case 2: $x \notin B$

Then $x \in A$ and $x \notin B$, so $x \in A-B$.

Therefore $x \in (A-B) \cup (A \cap B)$.

Thus, $A \subseteq (A-B) \cup (A \cap B)$.

Now we need to show $(A-B) \cup (A \cap B) \subseteq A$.

Pick any element $x \in (A-B) \cup (A \cap B)$.

Case 1: $x \in A-B$ ...

Then $x \in A$ and $x \notin B$

Therefore $x \in A$.

Case 2: $x \in A \cap B$ ...

Then $x \in A$ and $x \in B$

Therefore $x \in A$. ∎

Power Sets: Let A be any set

Its **power set** is the set whose **elements** are the **subsets** of A.
$P(A)$

Ex: $A = \{1, 2\}$.

$$P(A) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}.$$

Ex: $A = \{1,2,3\}$

$$P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

Exercise: Suppose A is a set with n elements. $(n \geq 0)$.
Then $P(A)$ has 2^n elements.

$A = \emptyset$
$P(A) = \{\emptyset\}$.

Proof: Induction on n. The statement to prove is

$Q(n)$: "For every set A of size n,
$P(A)$ has size 2^n."

<u>Base Case</u>: Suppose $n=0$.

There's only one zero-element set, $A=\emptyset$.

and $P(\emptyset)$ has size 1.

<u>Inductive Step</u>: Suppose A is a set of size $n+1$.

Call its elements $\{a_1, a_2, a_3, \dots, a_{n+1}\}$

We will prove there are 2^n subsets which don't contain a_{n+1}, and 2^n subsets which do contain a_{n+1}.

By the inductive hypothesis, the set $\{a_1, a_2, \dots, a_n\}$ has 2^n distinct subsets.

Call them $S_1, S_2, \dots, S_{2^n}$.

Then the set $\{a_1, a_2, \dots, a_{n+1}\}$ has subsets

$$S_1, S_2, \dots, S_{2^n} \quad \text{and} \quad S_1 \cup \{a_{n+1}\}, S_2 \cup \{a_{n+1}\}, \dots, S_{2^n} \cup \{a_{n+1}\}.$$

These sets are all distinct, any every subset of $\{a_1, \dots, a_{n+1}\}$ is in either the first list (if it doesn't contain a_{n+1}) or the second (if it does contain a_{n+1}).

This is a total of 2^{n+1} sets. So $P(A)$ has 2^{n+1} elements, as desired. ∎